GLOBAL UNIQUENESS OF A CLASS OF MULTIDIMENSIONAL INVERSE PROBLEMS

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Uniqueness theorems for multidimensional inverse problems have at present been obtained mainly in classes of piecewise analytic functions and similar classes or locally (see [1]–[7], [15] and the literature cited there). Moreover, the technique of investigating these problems has, as a rule, depended in an essential way on the type of the differential equation. In this note a new method of investigating inverse problems is proposed that is based on weighted a priori estimates. This method makes it possible to consider in a unified way a broad class of inverse problems for those equations \( Pu = f \) for which the solution of the Cauchy problem admits a Carleman estimate of the type considered in [8]–[11]. The theorems of §1 were proved by M. V. Klibanov and those of §2 by A. L. Buhgeim. They were obtained simultaneously and independently.

We consider the following inverse problems.

1. Determine functions \( u(x, t) \), \( a(x) \), and \( f(x) \) from the conditions

\[
\begin{align*}
(1) & \quad u_t = Lu, \quad x \in B^n, \quad t > 0; \\
(2) & \quad u(x, T) = F(x), \quad D^2 F(x) > 0, \quad T = \text{const} \\
(3) & \quad u(x, 0) = f(x),
\end{align*}
\]

where \( L = L_{\alpha, \beta} \) is a uniformly elliptic operator of second order in \( B^n \), and \( F \) and \( a(x) \), \( \alpha \neq \alpha_0 \), are given functions. Let \( T_0 \) be any number such that \( T_0 > T > 0 \), let \( \Omega = B^n \times (0, T_0) \), let \( G \) be a domain in \( B^n \), and let \( H^{k+\beta}(B^n) \) and \( H^{2k+\beta, k+\beta/2}(\Omega) \) (\( k \) is an integer, \( \beta \in (0, 2) \)) be the Hölder spaces [12].

**Theorem 1.** Suppose that in (1)–(3) \( a(x) \), \( f \in H^{2+\beta}(B^n) \), the function \( a(x) \) is known for \( x \in G \), and \( a(x) \in C^\infty(\Omega) \), \( |a(x)| \leq 2 \). Then there is not more than one vector-valued function \( (u, a_\alpha(x), f) \), \( u \in H^{2+\beta, 2+\beta/2}(\Omega) \), satisfying conditions (1)–(3).

The analyticity in \( t \) of the function \( u \) is used in the proof of Theorem 1 (see [13]). If the function \( a(x) \) is not given in the domain \( G \), then Theorem 1 is not true in general [14].

1.2. Determine functions \( u(x, t) \) and \( a(x) \) from the conditions

\[
\begin{align*}
(4) & \quad u_{tt} + Lu = 0, \quad x \in B^n, \quad t \in (0, T); \\
(5) & \quad u(x, 0) = f_1(x), \quad u_t(x, 0) = f_2(x);
\end{align*}
\]

where \( f_1, f_2, \) and \( a(x) \), \( \alpha \neq \alpha_0 \), are given functions.
THEOREM 2. Suppose that in (4) and (5) the function $a_{a_0}$ is known for $x \in G$ and $f_1,
$ $a_\alpha \in C^2(R^n)$, $|\alpha| \leq 2$, with $D^\alpha f_1(x) \neq 0$ for all $x \in R^n \setminus G$. Then there is not more than one
pair of functions $(u, a_{a_0}), u \in C^3(R^n \times [0, T])$, that is a solution of problem (4), (5).

1.3. Determine functions $u$ and $b$ from the conditions

$$\Delta u - \partial_y u = \sum_{|\alpha| \leq 1} a_\alpha(x, y) D^\alpha u + b(x, y') f(x, y),$$

$x \in \Omega_1 \subset R^n$, $y \in \Omega_2 \subset R^m$, $m \geq 1$, $y' = (y_1, y_2, \ldots, y_m-1)$;

$u(x, y', 0) = \psi_1(x, y')$, $u_{y'}(x, y', 0) = \psi_2(x, y')$;

$$u \bigg|_{x \in \partial \Omega_1} = F_1(x, y), \quad \frac{\partial u}{\partial n} \bigg|_{x \in \partial \Omega_1} = F_2(x, y).$$

Here $\Omega_1 = \{x \in R^n: |x| < r_1\}$ and $\Omega_2 = \{y \in R^m: |y| < r_2, y_m > 0\}$, $r_2 > r_1$; $n$ is the
unit normal to $\partial \Omega$; $\Delta$ is the Laplace operator (in the variables $x$ or $y$ respectively); and $f, \psi_i,
F_i$, and $a_\alpha, i = 1, 2$, $|\alpha| \leq 1$, are given functions. We set

$$B = \Omega_1 \times \Omega_2, B' = B \cap \{y_m = 0\}, \quad Z_0 = B' \times (0, \delta),$$

$$\omega_1 = \{(x, y)|x| = r_1, \quad |y| < r_1\}, \quad \omega_2 = \{(x, y)|x|^2 - |y|^2 = 0\}.$$

Let $\Omega \subset B$ be the domain enclosed between the hyperplane $y_m = 0$ and the hyperplanes
$\omega_1$ and $\omega_2$; let $M = C^2(\Omega) \cap C^1(\Omega)$. Theorem 3. Let $a_\alpha \in C(B)$, $|\alpha| \leq 1$, and suppose that $f \in C^2(Z_0)$ for some $\delta > 0$, and $f(x, y', 0) \neq 0, (x, y') \in B'$. Then there is not more than one vector-valued function
$(u, b) \in M \times C(B')$ satisfying conditions (6)--(8).

2. In this section $\Omega$ is a bounded domain in $R^n$ with a boundary of class $C^\infty$ which is
situated locally on one side of $\partial \Omega$.

2.1. We consider the two boundary value problems:

$$u_{rr} + L_j u_j = 0, \quad x \in \Omega, \quad t \in [0, T];$$

$$u^i(x, 0) = g(x), \quad u_{rj}(x, 0) = 0, \quad x \in \Omega;$$

$$\frac{\partial u^j}{\partial n} \bigg|_{r \times [0, T]} = 0, \quad j = 1, 2, \quad \Gamma = \partial \Omega.$$
in $\Omega \times [0, T]$ are strongly pseudoconvex in the sense of [8] with respect to the operator $D^2_t - L_1$.

If in (9) we replace $u_t$ by $-\psi_t$ (the parabolic case) and condition (10) by $u'(x, T') = g(x), T' \in (0, T)$, then for this inverse problem there is also an analogue of Theorem 4.

2.2. Suppose that $\phi$ is a real function of class $C^m(\Omega)$ with $|\phi(x)| + |\nabla \phi(x)| \neq 0, x \in \Omega$; let $\Omega_2 = \Omega \cap \{\phi > 0\}$. We suppose that $\Gamma_0 = \partial \Omega \setminus \{\phi = 0\}$ and $\Sigma \subset \Omega$ are hypersurfaces of class $C^m$.

We consider the problem of determining functions $u, f \in C^m(\Omega_2)$ from the conditions

$$\begin{align*}
(12) \quad P u &= f, \quad Q f = g; \\
(13) \quad pu &= (P_0 u, p_1 u, \ldots, p_{m-1} u) = h_1, \quad qu = (q_0 u, q_1 u, \ldots, q_{e-1} u) = h_2.
\end{align*}$$

Here $P$ and $Q$ are linear differential operators or order $m$ and $l$ respectively with coefficients in $C^m(\Omega)$;

$$\begin{align*}
p_j u &= \frac{\partial^j u}{\partial n^j} \bigg|_{\Gamma_0}, \\
q_j u &= \frac{\partial^j u}{\partial n^j} \bigg|_{\Sigma},
\end{align*}$$

$n$ is the unit normal to $\Gamma_0$ or $\Sigma$ respectively; and the function $g$ and the vector-valued functions $h_1$ and $h_2$ are given.

We say that the operators $P$ and $Q$ essentially commute if the order of the operator $[P, Q] = PQ - QP$ is equal to $m + l - 2$. We assume that the following estimates hold for $P$ and $Q$ for all $\tau \geq \tau_0$:

$$\begin{align*}
(14) \quad \forall u \in C^m(\Omega_0), \quad \tau \|u\|_{r,m-1}^2 + \frac{\alpha}{\tau} \|u\|_{r,m}^2 &\leq C \|Pu\|_{r,0}^2; \\
(15) \quad \forall u \in C^m(\Omega_0), \quad qu = 0 \quad \tau^s \|u\|_{r,s+1}^2 &\leq C_1 \|Qu\|_{r,s}^2,
\end{align*}$$

$s = m - 1, m$, where the constants $C, C_1 > 0, \alpha \geq 0$, and $\alpha$ do not depend on $u$ or $\tau$, and

$$\|u\|_{r,m}^2 = \sum_{|\alpha| \leq m} \int |D^\alpha u|^2 e^{2\tau \phi} dx.$$ 

**Theorem 5.** Suppose that the operators $P$ and $Q$ essentially commute $\alpha > -1$ or $\alpha = 1$, and $8CC_1 \|P, Q\|_{L(W^{l+m-1, w_0})} < a$ ($W^s = W^s(\Omega_0)$ is the Sobolev space of order $s$). If the hypersurface $\Gamma_0$ is characteristic relative to $Q$, then the solution of problem (12), (13) under conditions (14) and (15) is unique.

**Remark 2.** In applications usually $l = 1, \Sigma = \Omega_0 \cap \{x_n = 0\}$, and the leading part of the operator $Q$ is $D_n$. In this case for the validity of the estimate (15) with $\alpha = 0$ it suffices that $\varphi(x', x_n) \leq \varphi(x', 0), x' = (x_1, \ldots, x_{n-1})$. If $\varphi_{x_n}(x', 0) = 0$ and $\varphi_{x_n x_n} < 0$, then (15) holds for $\alpha = \frac{1}{2}$. Without the condition $qu = 0$ the estimate (15) is not true.

Questions of the uniqueness and stability of a broad circle of inverse coefficient problems reduce to the problems considered in §§ 1.3 and 2.2. Theorems 1–4 essentially follow from Theorem 5. Theorem 4 and theorems similar to it (see Remark 1) can also be proved by the method of §1.2.

The operators $P$ and $Q$ in Theorem 5 may also be integrodifferential operators, and their coefficients may be matrices. The smoothness assumptions can be considerably relaxed. Theorem 5 has an analogue in anisotropic Sobolev spaces.

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